TOPOLOGY FROM INTERIORS

by

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§ 1. Introduction

In the beginning of Hocking and Young [2], the fundamental structure of a topological space on a set X is determined by whether or not every point of X is a limit point of a subset of X. Based on this concept, the primitive notions open set (see Hocking and Young [2] p. 5), neighborhood (see Wallace [10] p. 14) and closure (see Pontrjagin [7] p. 26) are generally used to introduce a topology on X. Also there are some other ways, such as closed set, base and subbase, etc. Although the latter are not so often used to form the initial definition of a topology, their equivalent inter-relation with the former for defining a topology on X have been discussed intensively in many books (see Aleksandrov [1] p. 4-6; Mendelson [5] p. 84-99; Simmons [9] p. 97-102; Yang [11] p. 11-14).

This paper is devoted to define a topology by the interior of a subset E of X first. The main work of this paper is to show the equivalent inter-relation between the topology defined by interior and that defined by open set, neighborhood and closure in Theorems I, II, III, respectively. For this purpose, we do not want to discuss any properties which do not concern our theorems.

Most of the symbols used in this paper are the same as those given by Kelley [3] and Yang [11].

§ 2. Motivation and definition

The fundamental relation between the limit point and the interior of a set has been shown in Kelley [3] p. 44, i.e., that the set of all points of a set E which are not limit points of X-E (the complement of E) is precisely the interior E° of E. From this concept, the interior E° of a set E can be determined entirely by the limit points. Conversely, we can formulate the limit points of E in terms of interior. If the point x does not belong to E, then x is a limit point of E iff $x \notin (X\text{-}E)^\circ$. However in case $x \in E$, this criterion is not sufficient, since x can be an isolated point of E. But if $x \in E$ and is at the same time a limit point of E, then x is a limit point of $(E-x)$, i.e., $x \notin [(X-E) \cup x]^\circ$. This condition is sufficient. Moreover, $E=E-x$ when $x \notin E$. Hence it follows that x is a limit point of E iff $x \notin [(X-E)\cup x]^\circ$.

This conclusion and the duality between the "closure" and "interior" encourage us to give an alternative definition of topological space from interior.

Definition 1. Let X be a set. To every subset E of X there corresponds a subset E° which is called the interior of E. The family $\{E^\circ\}$ of all interiors on X satisfying the following conditions:

I−1. $E^\circ \subset E$;

I−2. $X^\circ = X$;

I−3. $E^\circ_\alpha \cap E^\circ_\beta = (E_\alpha \cap E_\beta)^\circ$;

I−4. $(E^\circ)^\circ = E^\circ$,

is called a topology on X. A topological space $(X, \{E^\circ\})$ consists of the set X and the topology $\{E^\circ\}$. The elements of X are called the points of $(X, \{E^\circ\})$.

Example 1. The most intuitive topology is the ordinary interior of an interval of real numbers. That is, let X be the set of all real numbers. The interior of any interval is the set without the endpoints of this interval. It can be verified that the family $\{E^\circ\}$ of such interiors is a topology. Then $(X, \{E^\circ\})$ forms a topological space.

Example 2. Let X be a set. We define the interior E° of a subset E of X as $E^\circ = E$. Then $(X, \{E^\circ\})$ forms a topological space.

A topological space defined in this way is called discrete topological space.

Example 3. Let X be an infinite set. We define the interior E° of a subset E of X as: $E^\circ = E$, if $(X-E)$ is finite and $E^\circ = \emptyset$ (the empty set), if $(X-E)$ is infinite.

Conditions I−1, I−2, and I−4 of Definition 1 are easily seen to be satisfied. To check I−3, we use the De Morgan formula

$$X-(E_\alpha \cap E_\beta) = (X-E_\alpha) \cup (X-E_\beta).$$

Example 4. Let X be the three−point set $\{a, b, c\}$. We define the interior E° of all subsets of X to be $\emptyset^\circ = \emptyset$, $\{a\}^\circ = \{a\}$, $\{b\}^\circ = \emptyset$, $\{c\}^\circ = \emptyset$, $\{a, b\}^\circ = \{a, b\}$, $\{a, c\}^\circ = \{a, c\}$, $\{b, c\}^\circ = \emptyset$ and $X^\circ = X$. It is clear that $(X, \{E^\circ\})$ forms a topological space.

Example 5. Let X be the set of all rational numbers. We define the interior of a subset $E = \{x: a \leqq x \leqq b$ for some rational numbers a, b$\}$ to be $E^\circ = \{x: \check{a} \leqq x \leqq \hat{b}$ for the largest integer $\hat{b} \in E$ which is $\leqq b$ and the smallest integer $\check{a} \in E$ which is $\geqq a\}$. If the infinities $\pm\infty$ of the real numbers are taken as integers, then the family $\{E^\circ\}$ is a topology on X.

§3. Open sets

Definition 2. A set U in a topological space $(X, \{E^\circ\})$ is called an open set iff $U^\circ = U$.

Lemma 3.1. Every member E° of the topology $\{E^\circ\}$ is an open set.

(Proof) $(E^\circ)^\circ = E^\circ$ (by I−4).

Lemma 3.2. Let $\{U\}$ be a family of all open subsets of a topological space $(X, \{E^\circ\})$. Then (1) X and $\emptyset$ are in $\{U\}$; (2) the union of any number of members of $\{U\}$ is in $\{U\}$; (3) the intersection of a finite members of $\{U\}$ is in $\{U\}$.

(Proof) (1) By I−2, $X^\circ = X$. Thus X is in $\{U\}$. By I−1, $\emptyset^\circ \subset \emptyset$. Since for any subset E of X, $\emptyset \subset E$. Hence $\emptyset^\circ = \emptyset$ and thus $\emptyset$ is in $\{U\}$.

(2) Let us first observe that if for any two subsets B and D of X such that

$B \subset D$. Then $B^\circ \subset D^\circ$ since $B^\circ = (B \cap D)^\circ = B^\circ \cap D^\circ$ by I−3. Suppose that $\{V\}$ is a family of any number of open set and $D = \cup\{V: V \in \{V\}\}$. Then every V is contained in D and therefore $V^\circ \subset D^\circ$, and thus $D^\circ \supset \{V^\circ: V \in \{V\}\} = \cup\{V: V \in \{V\}\} = D$. But $D^\circ \subset D$, it follows that $D^\circ = D$ and hence D is an open set.

(3) Let U_α and U_β be two open sets. Then $U_\alpha \cap U_\beta = U^\circ_\alpha \cap U^\circ_\beta = (U_\alpha \cap U_\beta)^\circ$. Thus $U_\alpha \cap U_\beta$ is an open set. By induction this is true for any finite number.

Lemma 3.3. Let E be a subset of X. Then the interior E° of E is the largest open set contained in E (see Kelley [3] P.44; Lefachetz [4] P. 32).

(Proof) Let $\{U\}$ be a family of all open subsets of the topological space $(X, \{E^\circ\})$ and let $\{V\} = \{V: V \in \{U\}, V \subset E\}$. By Lemma 3.1, E° is an open subset of E, and consequently $E^\circ \in \{V\}$. This implies that $E^\circ \subset \cup\{V: V \in \{V\}\}$. On the other hand, since $V \subset E$ for all $V \in \{V\}$, therefore $\cup\{V: V \in \{V\}\} \subset E$, and so $\cup\{V \in \{V\}\} = (\cup\{V: V \in \{V\}\})^\circ \subset E^\circ$. Hence $E^\circ = \cup\{V: V \in \{V\}\}$. That is E° is the largest open subset of E. The proof of this lemma is completed.

Theorem I. Let X be a set and let $\{B\}$ be a family of subsets of X having the properties (1), (2), (3) of lemma 3.2, i.e.,

0−1. $X \in \{B\}$, $\emptyset \in \{B\}$;

0−2. $\cup\{B_i: B \in \{B\}, i \in I$ for any index set $I\} \in \{B\}$;

0−3. $\cap\{Bi: B \in \{B\}, i \in I$ for a finite index set $I\} \in \{B\}$.

Then there is one and only one way of making X into a topological space so that $\{B\}$ is the family of all open sets of X.

(Proof) Part 1. It follows from Lemma 3.2 and Lemma 3.3 that there is only one way to define the interior E° of any subset E of X by the given family $\{B\}$. Namely, we have to let

$$E^\circ = \cup\{B: B \in \{B\}, B \subset E\}. \qquad \text{(A)}$$

Part 2. Now we must show that the family $\{E^\circ\}$ defined in (A) satisfies the four conditions of Definition 1.

We omit the easy proofs of the conditions I−1 and I−2.

To prove the condition I−3, we observe from (A) and 0−2 that $E^\circ \in \{B\}$ and E° is the largest member of $\{B\}$ ccontained in E. Let E_α and E_β be two subsets of X. Then $E^\circ_\alpha \cap E^\circ_\beta \subset E_\alpha \cap E_\beta$ and $E^\circ_\alpha \cap E^\circ_\beta \in \{B\}$ by 0−3. Hence $E^\circ_\alpha \cap E^\circ_\beta \subset (E_\alpha \cap E_\beta)^\circ$. On the other hand. we let B be a member of $\{B\}$ such that $B \subset (E_\alpha \cap E_\beta)$. Then both $B \subset E_\alpha$ and $B \subset E_\beta$, therefore $B \subset E^\circ_\alpha$ and $B \subset E^\circ_\beta$, and hence $B \subset E^\circ_\alpha \cap E^\circ_\beta$. Since B is an arbitrary member of $\{B\}$ contained in $E_\alpha \cap E_\beta$, i.e., $B \subset (E_\alpha \cap E_\beta)^\circ$, it follows that $(E_\alpha \cap E_\beta)^\circ \subset E^\circ_\alpha \cap E^\circ_\beta$. Hence $E^\circ_\alpha \cap E^\circ_\beta = (E_\alpha \cap E_\beta)^\circ$.

Let E° be the interior of E defined in (A). Then, as we have noted above, E° is the largest member of $\{B\}$ contained in E. Similarly, $(E^\circ)^\circ$ is the largest member of $\{B\}$ contained in E°. Since $E^\circ \in \{B\}$, so that $(E^\circ)^\circ = E^\circ$, and the condition I−4 is satisfied.

As we have shown above, $(X, \{E^\circ\})$ forms a topological space.

Part 3. The next thing is to show that, when X is made as above into a topological space, the open sets are precisely those sets which belong to the given family {B}. Let B∈{B}. Then B contains at least one member of {B}, namely B itself. And so $B^\circ=B$. By Definition 2, B is an open set. Conversely, let U be any open set of X. Since $\emptyset \in$ {B}, by 0−1, there is a non-empty subfamily of {B} contained in U whose union is U° by (A). But $U^\circ=U$ since U is open, and $U^\circ\in$ {B}, by 0−2. Hence U∈ {B}. Combining these, it follows that the family {U} of open sets concides with the family {B}. Thus the theorem is proved.

This theorem enables us to define the same topological space by means of open sets rather than by means of interiors. The definition of a topological space in terms of open sets is used quite generally, because from the logical point of view, it is simpler than that of interiors. In fact, many concepts can be made and theorems proved much more simply and elegantly on a topological space in terms of open sets than in terms of interiors. But from the practical point of view, to verify the conditions 0−1 to 0−3 is not so easy as to verify the conditions I−1 to I−4 from a previousely defined family with which we want to make a set X into a topological space.

§4. Neighborhoods

Definition 3. A set N in a topological space (X, $\{E^\circ\}$) is called a neighborhood of a point x of X iff the interior N° of N contains x. The family of all neighborhoods of x is denoted by $\{N_x\}$. A neighborhood of a point needs not be an open set (see Kelley [3] P. 38).

Lemma 4.1. Let $\{N_x\}$ be the family of all neighborhoods of x in a topological space (X, $\{E^\circ\}$). Then $\{N_x\}$ is a non-empty family.

(Proof) Since $X^\circ=X$ by I−2, $X\in\{N_x\}$, i.e., the set X is the neighborhood of all its points.

Lemma 4.2. Let $\{N_x\}$ be the family of all neighborhoods of x in a topological space (X, $\{E^\circ\}$). Then (1) if N_x is a neighborhood of x, then $x\in N_x$; (2) any subset of X containing a neighborhood of x is itself a neighborhood of x; (3) if N_x and M_x are two neighborhoods of x, so is $N_x\cap M_x$; (4) if N_x is a neighborhood of x, there is a neighborhood M_x of x such that N_x is a neighborhood of every point of M_x.

(Proof) (1) $x\in N^\circ_x\subset N_x$ by Definition 3 and I−1.

(2) Let $N_x\in\{N_x\}$ and $N_x\subset M\subset X$. Then $x\in N^\circ_x$, $N^\circ_x\subset M^\circ$, and therefore $x\in M^\circ$. Hence $M\in\{N_x\}$, i.e., M is a neighborhood of x.

(3) Since $N_x\in\{N_x\}$ and $M\in\{N_x\}$, then $x\in N^\circ_x$ and $x\in M^\circ_x$. This follows that $x\in N^\circ_x\cap M^\circ_x=(N_x\cap M_x)^\circ$, by I−3. Henece $N_x\cap M_x\in\{N_x\}$.

(4) If $N_x\in\{N_x\}$, then, by Definition 3, $x\in N^\circ_x$.

Since $(N^\circ_x)^\circ=N^\circ_x$ by I−4, N°_x itself is a member of $\{N_x\}$. Moreover, $N^\circ_x\in\{N_Y\}$ for every $y\in N^\circ_x$. So $M_x=N^\circ_x$ is the desired neighborhood of x.

Lemma 4.3. A subset B of X is a member of $\{E^\circ\}$ iff B is a neighborhood of each of its points.

(Proof) The necessity of this lemma has been shown in proving (4) of Lemma 4.2. We now prove its sufficiency. If $B\in\{N_x\}$ for every $x\in B$, then, by Definition 3. $x\in B^\circ$. Therefor $B\subset B^\circ$. But $B\supset B^\circ$ and hence $B=B^\circ$. Thus $B\in\{E^\circ\}$.

Theorem II. Let X be a set. To each point x of X there corresponds a non-empty family $\{H_x\}$ of subsets of X. If $\{H_x\}$ satisfies the properties of Lemma 4.2, i.e.,

N-1. $x \in H_x$ for every $H_x \in \{H_x\}$;

N-2. $H_x \subset M \subset X$ for some $H_x \in \{H_x\}$, then $M \in \{H_x\}$;

N-3. $H_x \in \{H_x\}$ and $G_x \in \{H_x\}$, then $H_x \cap G_x \in \{H_x\}$;

N-4. $H_x \in \{H_x\}$, there is $G_x \in \{H_x\}$ such that $H_x \in \{H_y\}$ for every $y \in G_x$.

Then there is a unique topology $\{E^\circ\}$ on X such that the topological space $(X, \{E^\circ\})$ has $\{H_x\}$ as a family of all neighborhoods of the point x.

(Proof) Part 1. First we have to give a suitable definition of the interior to any subset of X in terms of the given family $\{H_x\}$. From Lemma 4.2 and Lemma 4.3, it is reasonable to define the interior E° of E as

$E^\circ = \{x: H_x \subset E \text{ for some } H_x \in \{H_x\}\}$ ....(B)

Part 2. As E° defined in (B), the family $\{E^\circ\}$ will satisfy the conditions of Definition 1.

If $x \notin E$, the $H_x \not\subset E$ for every $H_x \in \{H_x\}$ by N-1, and $x \notin E^\circ$ by (B). Thus $E^\circ \subset E$ and I–1 is satisfied.

Since the family $\{H_x\}$ is non-empty and $H_x \subset X$ for every $H_x \in \{H_x\}$, hence $X^\circ = X$. The condition I–2 is satisfied.

Let $y \in E^\circ_\alpha \cap E^\circ_\beta$. Then there exist two members H_y and G_y of $\{H_y\}$ such that $H_y \subset E_\alpha$ and $G_y \subset E_\beta$. Thus $H_y \cap G_y \subset E_\alpha \cap E_\beta$. By N-3, $H_y \cap G_y \in \{H_y\}$. Clearly, $y \in (E_\alpha \cap E_\beta)^\circ$ and therefore $E^\circ_\alpha \cap E^\circ_\beta \subset (E_\alpha \cap E_\beta)^\circ$. On the other hand, $(E_\alpha \cap E_\beta)^\circ$

$= \{x: H_x \subset (E_\alpha \cap E_\beta) \text{ for some } H_x \in \{H_x\}\}$

$= \{x: H_x \subset (E_\alpha \cap E_\beta) \subset E_\alpha \text{ for some } H_x \in \{H_x\}\} \cap \{x: H_x \subset (E_\alpha \cap E_\beta) \subset E_\beta \text{ for some } H_x \in \{H_x\}\}$

$\subset \{x: H_x \subset E_\alpha \text{ for some } H_x \in \{H_x\}\} \cap \{x: H_x \subset E_\beta \text{ for some } H_x \in \{H_x\}\}$

$= E^\circ_\alpha \cap E^\circ_\beta$.

That is $(E_\alpha \cap E_\beta)^\circ \subset E^\circ_\alpha \cap E^\circ_\beta$. Hence $E^\circ_\alpha \cap E^\circ_\beta = (E_\alpha \cap E_\beta)^\circ$ and thus I–3 is satisfied.

Suppose that $x \notin (E^\circ)^\circ$. This implies that $H_x \not\subset E^\circ$ for every $H_x \in \{H_x\}$. By N-4, to each $H_x \in \{H_x\}$ there exists $G_x \in \{H_x\}$ such that $H_x \in \{H_y\}$ whenever $y \in G_x$. Let $y \in G_x$, but $y \notin E^\circ$. Then $H_y \not\subset E$ for every $H_y \in \{H_y\}$ and thus $H_x \not\subset E$. It follows that $x \notin E^\circ$ and hence $E^\circ \subset (E^\circ)^\circ$. On the other hand we have shown that $(E^\circ)^\circ \subset E^\circ$. Hence $(E^\circ)^\circ = E^\circ$, i.e., condition I-4 of Definition 1 is satisfied.

So X is made into a topological space $(X, \{E^\circ\})$.

Part 3. Now we shall prove that the family $\{H_x\}$ to x will be the family $\{N_x\}$ of all neighborhoods of x on the topological space $(X, \{E^\circ\})$.

Let $H_x \in \{H_x\}$. Then $x \in H_x$ (by N-1) and there exists $G_x \in \{H_x\}$ such that $x \in G_x \subset H_x$ (by N-4). According (B), $x \in H^\circ_x$, and hence $H_x \in \{N_x\}$ by Definition 3. On the other hand, suppose that $N_x \in \{N_x\}$, then $x \in N^\circ_x$ (by Definition 3) and there exists $H_x \in \{H_x\}$ such that $x \in H_x \subset N_x$ (by (B)). Thus $N_x \in \{H_x\}$ by N-2. Hence $\{H_x\} = \{N_x\}$.

Part 4. In looking back to Lemma 4.3, we can easily find that it is equivalent to (B). Hence there is a unique topology $\{E^\circ\}$, as we have made above, with which the topological space $(X, \{E^\circ\})$ has the family $\{H_x\}$ as the neighborhoods of x. The proof of Theorem II is thus completed.

This theorem shows that we can define a topological space from neighborhoods instead of interiors. (see Mendelson [5] p. 89; Wallace [10] p.14).

The definition of a topological space in terms of neighborhoods is probably the most convenient from the intuitive point of view, because it gives a common sense notion of nearness.

§5. Closure

Definition 4. A point x in a topological space $(X, \{E^\circ\})$ is a point of closure of a subset K of X iff any E° of $\{E^\circ\}$ containing x contains a point of K. The closure of K will be denoted by K^-.

Lemma 5.1. Let $\{K^-\}$ be a family of all closures in the topological space $(X, \{E^\circ\})$. Then (1) the closure of any set K contains the set K; (2) the closure of the empty set $\emptyset$ is the empty set $\emptyset$ itself; (3) the union of closures of two sets is equal the closure of the union of these two sets: (4) the closure of the closure of any set is equal to the closure of the set.

(Proof) (1) Let $x \in K$. Then whenever $x \in E^\circ \in \{E^\circ\}$, $E^\circ \cap K \neq \emptyset$. Moreover, there is at least one $E^\circ \in \{E^\circ\}$, namely $E^\circ = X$, such that $x \in X$ and $X \cap K = K \neq \emptyset$. Thus $x \in K^-$. This implies that $K \subset K^-$.

(2) We omit the easy proof.

(3) Let K^-_α and K^-_β be two members of $\{K^-\}$. Then

$x \in K^-_\alpha \cup K^-_\beta \Longrightarrow$ whenever $x \in E^\circ \in \{E^\circ\}$, $E^\circ \cap K_\alpha \neq \emptyset$ or $E^\circ \cap K_\beta \neq \emptyset$.

$\Longrightarrow$ whenever $x \in E^\circ \in \{E^\circ\}$, $E^\circ \cap (K_\alpha \cup K_\beta) \neq \emptyset$.

$\Longrightarrow x \in (K_\alpha \cup K_\beta)^-$. Now

$x \notin K^-_\alpha \cup K^-_\beta \Longrightarrow$ there exist two members E°_α and E°_β of $\{E^\circ\}$ such that $x \in E^\circ_\alpha$, $x \in E^\circ_\beta$ and $E^\circ_\alpha \cap K_\alpha = \emptyset$, $E^\circ_\beta \cap K_\beta = \emptyset$.

$\Longrightarrow x \in (E^\circ_\alpha \cap E^\circ_\beta) = (E_\alpha \cap E_\beta)^\circ$ (by I-3) and $(E_\alpha \cap E_\beta)^\circ \cap (K_\alpha \cup K_\beta) = \emptyset$.

$\Longrightarrow x \notin (K_\alpha \cup K_\beta)^-$. Hence $(K_\alpha \cup K_\beta)^- = K^-_\alpha \cup K^-_\beta$.

(4) Let $x \in (K^-)^-$. Then

$x \in (K^-)^- \Longrightarrow$ whenever $x \in E^\circ \in \{E^\circ\}$, $E^\circ \cap K^- \neq \emptyset$.

$\Longrightarrow$ there is some $y \in E^\circ$ such that $y \in E^\circ$ and $y \in K^-$.

$\Longrightarrow E^\circ \cap K \neq \emptyset$ (by Definition 4)

$\Longrightarrow x \in K^-$. Thus $(K^-)^- \subset K^-$. But $K^- \subset (K^-)^-$, hence $(K^-)^- = K^-$.

Lemma 5.2. Let D be any subset of X. Then $X - D^\circ = (X-D)^-$ and $X - D^- = (X-D)^\circ$ (see Newman [6] P.30).

(Proof) Let x be any point of $X - D^\circ$. Then

$x \in X - D^\circ \Longleftrightarrow x \notin D^\circ$

$\Longleftrightarrow$ there is no $E^\circ \in \{E^\circ\}$ such that $x \in E^\circ$ and $E^\circ \subset D$ (by Lemma3.1 and Lemma3.3)

$\Longleftrightarrow$ any $E^\circ \in \{E^\circ\}$ contains x and $E^\circ \cap (x-D) \neq \emptyset$.

$\Longleftrightarrow x \in (X-D)^-$. Hence $X - D^\circ = (X-D)^-$.

The second half can be proved as

$(X-D^-) = X - [X-(X-D)]^- = X - [X-(X-D)^\circ]$ (by the first half) $= (X-D)^\circ$.

Theorem III. Let X be a set. To each subset G of X there corresponds a subset G^-. The family $\{G^-\}$ of all G^- in X satisfies the properties of Lemma 5.1, i.e.,

C−1. $G \subset G^-$;

C−2. $\emptyset = \emptyset^-$;

C−3. $G^-_\alpha \cup G^-_\beta = (G_\alpha \cup G_\beta)^-$;

C−4. $(G^-)^- = G^-$.

Then there is a unique topology $\{E^\circ\}$ on X such that the family $\{G^-\}$ is the family $\{K^-\}$ of all closures in the topological space $(X, \{E^\circ\})$.

(Proof) The proof of this theorem is similar to that of Theorem I and Theorem II, so we shall give a brief proof.

From Lemma 5.1 and Lemma 5.2, we shall define E° to each subset E of X in terms of the given family $\{G^-\}$ as

$$\{E^\circ = X - G^- : E = X - G,\ G^- \in \{G^-\}\} \cdots\cdots\cdots\cdots (C)$$

Then the conditions of Definition 1 will be satisfied.

Let E° be any member of $\{E^\circ\}$. Then

$E^\circ = (X-G)^\circ = (X-G^-) \subset (X-G)$ (by C−1) $= E$. Thus I-1 is satisfied.

Since $X = X - \emptyset$, $X^\circ = (X-\emptyset)^\circ = X - \emptyset^- = X - \emptyset$ (by C−2) $= X$. Thus I-2 is satisfied.

Let E°_α and E°_β be any two members of $\{E^\circ\}$. Then

$E^\circ_\alpha \cap E^\circ_\beta = (X-G_\alpha)^\circ \cap (X-G_\beta)^\circ = (X-G^-_\alpha) \cap (X-G^-_\beta) = X - (G^-_\alpha \cup G^-_\beta) = X - (G_\alpha \cup G_\beta)^-$ (by C−3) $= [X-(G_\alpha \cup G_\beta)]^\circ = [(X-G_\alpha) \cap (X-G_\beta)]^\circ = (E_\alpha \cap E_\beta)^\circ$.

Thus I-3 is satisfied.

Let E° be any member of $\{E^\circ\}$. Then $E^\circ = (X-G)^\circ$, thus

$(E^\circ)^\circ = [(X-G)^\circ]^\circ = (X-G^-)^\circ = X - (G^-)^- = X - G^-$ (by C−4) $= (X-G)^\circ = E^\circ$.

So that I-4 is satisfied.

As we have shown above, $\{E^\circ\}$ is a topology on X. It is easily seen that $\{E^\circ\}$ is the desired topology which makes the family $\{G^-\}$ as the all closures in the topological space $(X, \{E^\circ\})$. And this topology is unique (by Lemma 5.1, Lemma 5.2 and (C)).

We know from this theorem that the same topological space also can be defined by closures, if we can give a suitable definition to interior (see Kelley [3] p. 43; Patterson [7] p. 41; Pontrjagin [8] p. 26).

We see that the axioms I-1 to I-4 are dual to the axioms C-1 to C-4 in Boolean Algebra (see Newman [6] p. 7). If we consider the symbols "o" and "-" as the interior operator and closure operator respectively, then they are dual. It is due to this duality, we can give the definition of a topological space with I-1 to I-4 in §1.

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內域拓撲空間

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拓撲學 (Topology) 的發展奠定了近代數學的基礎。究竟如何在一集合 (set) X 上構作一拓撲而使 X 成一拓撲空間 (Topological Space 以下簡稱空間)？在 Hocking 及 Young〔2〕一書的開始即有一關於回答此問題的基本定義：『在任一集合 X 上有一拓撲即是能確定 X 中任一點是否爲 X 之任一子集合 (Subset) 之極限點 (Limit Point)』。此定義雖然對拓撲之構作上毫無具體的方法，但却指出一條簡明之路。凡能滿足此定義之諸概念 (Concept)，則此等概念之特性皆能適當的用來定義空間。於是吾人會以開集合 (Open set)，鄰域 (Neighborhood) 及閉包 (Closure) 來定義空間。此外尙有底 (Base)，子底 (Subbase) 及閉集合 (Closed set) 等方法，後者雖不常被用直接定義空間但其所定義之空間確能與前者所定義之空間合一，而彼此互應之關係亦常被討論。

因概念內域 (Interior) 亦能滿足上述定義，本文由內域之性質定義空間，進而在定理Ⅰ，Ⅱ，Ⅲ中分別討論由內域定義之空間與開集合，鄰域及閉包所定義之空間彼此互應而合一。